equal to zero." In this memoir Mr. Earnshaw points out the falacies in the proofs that have been given of the equation $\cos\infty = 0$. In the same vol., De Morgan had, at page 191, given reasons why we should expect periodic functions, when indeterminate, to be represented by their mean values, and remarks that the indeterminate symbols, $\sin\infty$ and $\cos\infty$, are found in numberless cases to represent, each of them, 0, the mean value of both $\sin x$ and $\cos x$. Mr. J. W. L. Glaisher also discusses this question in the 5th volume of the first series of the *Messenger of Mathematics* with a view of determining the conditions under which these expressions may be taken equal to zero; or more generally, under which a periodic function may be assumed equal (when x is infinite) to its mean value, or φ being a rational function,

$$\varphi(\sin\infty, \cos\infty) = \frac{1}{2\pi}\int_0^{2\pi}\varphi(\sin x, \cos x)dx.$$

The paradox is not without interest inasmuch as the geometrical illustration renders it evident that the effective value of $du \div dx$ ought to coincide with its mean value.

In this connection I may also mention that the difficulty arising in the application of the theorem

$$\frac{f(x)}{\varphi(x)} = \frac{f'(x)}{f'(x)} \text{ to the example } \frac{x - \sin x}{x + \cos x},$$

when $f(x) = \infty$, $\varphi(x) = \infty$ when $x = \infty$, given in Bertrand's Calcul Differential, p. 476 (quoted in Rice and Johnson's Calculus, p. 114), disappears if we admit that $\sin\infty$ and $\cos\infty$ are each equal zero.

W. W. Johnson.

Annapolis, Md. Nov. 28, 1882.

GEOMETRICAL DETERMINATION OF THE SOLIDITY OF THE PARABOLA.

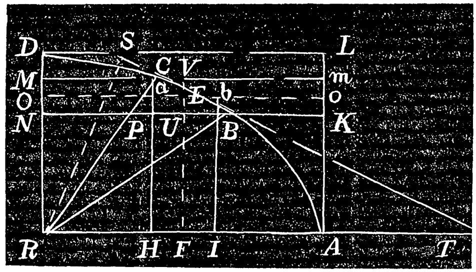
BY OCTAVIAN L. MATHIOT, BALTIMORE, MARYLAND.

Let AD be a parbola inscribed in the parallelogram $ALDR$. Suppose RCB to be one of an infinite number of inscribed triangles. Through C and B draw Mm and NK respectively parallel to RA, and draw CPH perpendicular to NK. Through E, the middle point of CB, draw the tangent ST to meet RS a perpendicular to it in S; also through E draw Oo parallel to Mm, and VUF perpendicular to RA.

From the similar triangles PCB and SRT we have

$$PC : CB :: RS : RT.$$

$$\therefore PC \times RT = RS \times CB.$$



The area of the triangle $RCB = \frac{1}{2}(CB\times RS) = \frac{1}{2}(PC\times RT)$.

Since $RT = RF + FT$, $PC.RT = PC.RF + PC.FT$. (1)

Since $RF = OE$, and FT being subtangent $= 2AF = 2Eo$; $\therefore PC\times RF = VU\times OE$, (2) and $PC\times FT = VU.2Eo$. (3)

From (1), (2) and (3) we get, by substitution,

$$\text{area } RCB = \tfrac{1}{2}(VU\times OE) + VU\times Eo \qquad (4)$$

If the triangle RCB be revolved about RA as an axis it will generate a volume $= \frac{2}{3}\pi.2EF.RCB$, or by substituting from (4),

$$= \tfrac{2}{3}\pi\times EF\times VU\times OE + \tfrac{2}{3}\pi\times 2EF\times VU\times Eo. \qquad (5)$$

But $2\pi.EF.VU =$ the area generated by revolving the line VU with radius EF around RA a an axis, which if multiplied by OE will equal the solidity of a hollow cylinder formed by revolving $MNUV$ around RA with radius EF. And $2\pi.EF.VU$, if multiplied by Eo will equal the solidity of the hollow cylinder formed by revolving $VUKm$ around RA, with radius EF. Hence the volume generated by the revolution of the trian. RCB around $RA = \frac{2}{3}$ of half the volume generated by $MNUV + \frac{2}{3}$ of the volume generated by $VUKm$, when revolved about the same axis RA. (6)

Since the bases of the inscribed triangles are infinitely small, the sum of the volumes generated by $MNUV$ equals the paraboloid, which call P; and the sum of the volumes generated by $VUKm$ equals the solid formed by the revolution of $DEAL$ about RA, which call S.

Hence by (6) we have for the sum of the volumes generated by RCB, or

$$P = \tfrac{2}{3}(\tfrac{1}{2}P + S);\ \therefore P = S.$$

Since $S+P =$ the volume of a cylinder circumscribed about the paraboloid, therefore the volume of a paraboloid is one-half the volume of its circumscribing cylinder, the axis of abscissas being the axis of revolution.

2. Let the parabola revolve about RD as an axis. Then the volume generated by the revolution of RCB about $RD = \frac{2}{3}\pi.2EO.RCB$. (7)

Now $2EO = 2AR - 2AF$ (8), and by (4) $RCB = \frac{1}{2}VU.OE + VU.Eo$, $= \frac{1}{2}VU(RA - FA) + VU.FA$ (9). Substitutin in (7) from (8) and (9) we get volume generated by the revolution of $RCB = \frac{2}{3}\pi(2RA - 2FA)\times [\frac{1}{2}VU(RA - FA) + VU.FA] = \frac{2}{3}\pi(2RA - 2FA)\frac{1}{2}(RA - FA)VU$

$$+\tfrac{2}{3}\pi(2RA - 2FA)FA.VU$$

$$= \tfrac{2}{3}\pi(RA^2 - 2RA.FA + FA^2)VU + \tfrac{2}{3}\pi(2RA.FA - 2FA^2)VU$$

$$= \tfrac{2}{3}\pi(RA^2 - FA^2)VU = \tfrac{2}{3}\pi.RA^2.VU - \tfrac{2}{3}\pi.Eo^2.VU. \qquad (10)$$

Now $\pi.RA^2$ is the area of the base of a circumscribing cylinder the axis of which coincides with, and whose length is equal to, RD, which call C.

The factor $\pi.Eo^2$ in (10) represents the area of a circle, radius Eo and center on the line AL, and when multiplied by VU, and summed for all positions of the triangle RCB it represents the volume generated by the revolution of the area $AEDL$ about AL as an axis, which volume call S'.

Substituting in (10) we have $P' = \frac{2}{3}C - \frac{2}{3}S'$. (11)

Prolong IB to b. The revolution of $HIab$ about LA generates the vol. $\pi.2FA.HI.EF$ (12); while the rev. of $VUKm$ produces $\pi.FA^2.VU$. (13)

Since $EF.HI = 2FA.VU$ (see ANALYST, Vol. IX, p. 107); therefore by substitution (12) becomes $2\pi.FA.2FA.VU = 4\pi.FA^2.VU$. (14)

Comparing (13) and (14) (as $Eo = FA$) we see that the volume denoted by (12) = four times the volume denoted by (13); and as this relation is constant it follows that the volume generated by the revolution of the area $AEDR$ about $DL = 4S'$; therefore the circumscribing cylinder $C = 5S'$, and consequently $S' = \frac{1}{5}C$. Substituting this value of S' in (11) we have

$$P' = \tfrac{2}{3}C - \tfrac{2}{15}C = \tfrac{8}{15}C.$$

[If x and y represent the coordinates of the point R on the axis of the parabola, then is $\frac{3}{5}x$ the distance of its center of gravity from the vertex, A; and by the theorem of Guldinus we have $\frac{2}{3}xy \times \frac{4}{5}\pi x = \frac{8}{15}\pi x^2 y =$ the vol. generated by the revolution of the parabola about its limiting ordinate, y, $= \frac{8}{15}$ of the volume of a cylinder whose radius is x and altitude y. This agrees with the above result so ingeneously deduced by Mr. Mathiot.—Ed.]

NOTE BY HENRY HEATON. — To integrate $\dfrac{dx}{(a+b\tan x)^n}$, differentiate $\dfrac{1}{(a+b\tan x)^{n-1}}$ and we get

$$-\frac{(n-1)b\sec^2 x\,dx}{(a+b\tan x)^n} = -\frac{(n-1)\,(a^2+b^2)dx}{b(a+b)\tan x)^n} + \frac{2a(n-1)dx}{b(a+b\tan x)^{n-1}} - \frac{(n-1)dx}{b(a+b\tan x)^{n-2}}.$$

Hence

$$\int\frac{dx}{(a+b\tan x)^n} = \frac{-b}{(n-1)(a^2+b^2)(a+b\tan x)^{n-1}} + \frac{2a}{a^2+b^2}\int\frac{dx}{(a+b\tan x)^{n-1}} - \frac{1}{a^2+b^2}\int\frac{dx}{(a+b\tan x)^{n-2}}$$

In like manner, differentiating $\dfrac{\tan x}{(a+b\sec x)^{n-1}}$, we get

$$\int\frac{dx}{(a+b\sec x)^n} = \frac{-b^2\tan x}{a(n-1)(a^2-b^2)(a+b\sec x)^{n-1}} + \frac{(3n-2)a^2-(n-1)b^2}{a(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-1}} - \frac{3n-1}{(n-1)\,(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-2}} + \frac{n}{a(n-1)\,(a^2-b^2)}\int\frac{dx}{(a+b\sec x)^{n-3}}.$$